1-1-1957

Some Aggregate Business Cycle Models

Hyman P. Minsky Ph.D.

Master

Some Aggregate Business Cycle Models

By

H. P. Minsky

Introduction

A large part of current business cycle theory consists of the construction of dynamic aggregate models which generate a time series of income. The aim of this activity is to construct a model which generates a time series whose characteristics are similar to observations of income. The various formulations of the dynamic income generating process that have been presented differ in many details. However, there is a large family of such models that are based upon the interaction of the accelerator and the multiplier relations. This family of models will be examined, starting with the simplest linear model and progressing through various modifications.

The simplest linear accelerator-multiplier model has serious defects. This model can generate only a limited number of types of time series, none of which are sufficiently like what has been observed to permit the acceptance of the simplest linear model. However by making additional specifications models which more closely conform to the observed patterns can be derived.

The emphasis in this paper is upon business cycle models. Essentially the same mechanism that is examined here can be used to construct both growth models and models which generate both cycles and growth. To facilitate the exposition the two problems are separated: only the construction of business cycle models #=(1) will be examined here.

In the first section the problems involved in constructing a useful framework for the analysis of business cycles will be stated. This will be followed by a discussion of the linear model. In the next sections alternative (though not necessarily mutually exclusive) modifications of the linear accelerator-multiplier model will be examined. In the concluding section it will be pointed out that the accelerator-multiplier hypothesis is useful because it can be modified in these alternative ways.

I. Problems in Using the Accelerator-Multiplier Framework

Although the search for a satisfactory formal model of the income generating process may seem like a mathematical game, the serious purpose of the effort is to discover an apparatus which can serve as a useful framework for the analysis of events as they occur. Such a framework is desired in the belief that it will make possible both the prediction and hence ultimately the control of events.

The accelerator-multiplier approach is both formal and abstract. It is formal in the sense that the solution is a mathematical consequence of the assumptions. It is abstract in two senses:

(1) The models directly generate a time series of only one variable, conveniently called income.

(2) The models have but a few parameters and the underlying behavior relations greatly simplify the behavior of the various classes of economic units.

In contrast to this approach is the one that examines the observed behavior of and the temporal relations among a multitude of time series: the more the merrier. It is assumed that from the examination of the many series a regular pattern of events will ultimately be revealed. This revealed pattern is to be both the predictive tool and the key to the correct analytical apparatus. The "way of working" of this approach is to prepare a catalogue of time series of raw data and of constructed indices, and, from this catalogue and economic history, to prepare a "consensus" series. The relations among the many series and between each series and the "consensus" series is examined, and a small number of series that consistently lead and conform to the consensus of all series is derived. These leading and conforming series become the indicators. Aside from this "correlation" no analytical relations between the various key series are derived and therefore no reason exists why the pattern in the future should be similar to that which was observed in the past. The predictive value of the results depends upon the crude assumption that there exists some <u>unknown</u> mechanism which generates the observations, and that this unknown mechanism does not change. That is, in spite of the growth and evolutionary attributes of the economy, it is naively assumed that the "future is like the past." Note that if this method predicts undesirable events there is nothing in the apparatus to indicate how the unwanted events can be avoided. Even if the indicator apparatus consistently leads to good predictions, as long as no behavioral basis for the lead and lag relationships are postulated and tested there are no control possibilities.

The one-dimensional set of models being studied here is based upon behavioral assumptions. The models are constructed on the basis of formal statements of how presumably significant groups react to the variables in the model. On the basis of these assumptions the formal model generates a time series of income. If this time series is sufficiently similar to observations, the model can be used as a predictive tool. If an unwanted event is predicted, the possibility exists that operations upon the world can affect the behavior of the significant groups. This would make the future different from that predicted and presumably the unwanted event would now not occur. Hence an analytical approach to the problem does hold out the possibility of achieving an operationally significant analysis of business cycles.

An economy is a complex evolving beast. Any formal apparatus is necessarily based upon a small number of relations which isolate and emphasize a limited number of attributes. For the formal model to be useful it is necessary to define how variations in attributes not contained in the model affect the workings of the model; that is, it is necessary to specify how events in that part of the world ignored in building the model (the major part of the world) are to be fed into the model. Obviously if the model is to be useful the specifications for most such events will be that they have no effect, however this specification is not true for all events not contained in the formalized reactions.

The simplest linear model leads to very limited and rigid results. For the accelerator-multiplier framework to be useful this rigidity, particularly with respect to the period and the amplitude of the time series generated, must be relaxed. Various ways of relaxing the specifications of, and of feeding excluded events into, the linear models will be examined. In the limited scope of this paper no exhaustive catalogue of how excluded events are to be fed into the model can be presented. However it will be shown that various modifications of the linear model yield cycles with the desired irregularity. ~~in the cycles~~.

II. The Simple Linear Homogeneous Model

An elementary accelerator-multiplier mechanism was presented by Samuelson (10). Its specifications are:

(A) Income of any date is defined as

$$Y_t = C_t + I_t \tag{1}$$

where Y_t is the income, C_t the consumption and I_t the investment of the $\underline{t}$th date.

(B) Consumption of any period depends upon the income of the previous period, that is

$$C_t = \alpha Y_{t-1} \tag{2}$$

where α (a constant) is the marginal (in this model equal to the average) propensity to consume and, $0 < \alpha < 1$.

(C) Investment of any period depends upon the realized change in income, that is

$$I_t = \beta\,(Y_{t-1} - Y_{t-2}) \tag{3}$$

where β (a constant) is the accelerator coefficient and $\beta > 0$. All lags are of the same length.

These specifications yield the second order linear homogeneous difference equation:

$$Y_t = (\alpha + \beta)\, Y_{t-1} - \beta Y_{t-2}. \tag{4}$$

Equation 4 is second order because the longest lag is two periods in length, it is linear because the sum of the powers of the variable, the dated income, in each element is one and it is homogeneous because if $Y_{t-1} = Y_{t-2} = 0$, then $Y_t = 0$, the equation passes through the origin. The object is to find a solution to equation 4, a solution being a relation that determines the income of any period as a function of the date of the period.

Assuming that α and β are known real numbers, it is obvious that once two successive incomes are known, repeated applications of equation 4 will generate the income of any date, future or past. To do this the following information is needed:

(1) The values of α and β.

(2) Two initial incomes.

The solution to a difference equation (that is, writing income as a function of time) does not economize on the information required, rather it makes for more efficient use of the same information. The solution equation is more efficient than the original difference equation in that:

(1) the income of any date can be determined without going through a tedious iterative process;

(2) the solution equation yields information on the nature of the time series generated by the difference equation without requiring the actual generation of the series.

The solution to equation 4 takes the form

$$Y_t = A_1 \mu_1^{\,t} + A_2 \mu_2^{\,t} \tag{5}$$

where μ_1 and μ_2 depend upon α and β and A_1 and A_2 depend upon the initial conditions. To derive the solution equation, X^t is substituted for Y_t in equation 4, yielding

$$X^t = (\alpha + \beta)\, X^{t-1} - \beta X^{t-2} \tag{6}$$

which can be written as

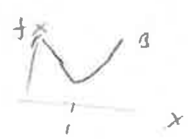
$$f(X) = X^2 - (\alpha + \beta)\, X + \beta. \tag{7}$$

Equation 7 is the characteristic equation of the second order linear homogeneous difference equation. As equation 7 is a quadratic equation, its solution (those values of x for which $f(x) = 0$) is easily determined. This solution yields two functions of α and β, designated by μ_1 and μ_2, which become the constants in (or roots of) the solution equation. These roots carry the information contained in the known α and β from the original difference equation to the solution equation.

Some knowledge of the roots μ_1 and μ_2 can be obtained by examining the characteristic equation, equation 7. The derivative with respect to x of equation 7 is

$$f^1(x) = 2x - (\alpha + \beta). \tag{8}$$

If $x < \frac{\alpha + \beta}{2}$ the derivative is negative, if $x > \frac{\alpha + \beta}{2}$ it is positive. Hence $f(x)$ is decreasing in the range $-\infty < x < \frac{\alpha + \beta}{2}$ and increasing in the range $\frac{\alpha + \beta}{2} < x < \infty$. The minimum point of $f(x)$ occurs when $x = \frac{\alpha + \beta}{2}$. If $f\left(\frac{\alpha + \beta}{2}\right) > 0$ the roots are conjugate complex numbers. Equations 7 and 8 yield the following table:



X	f(x)	f'(x)
0	$\beta > 0$	$-(\alpha + \beta) < 0$
1	$1 - \alpha > 0$	$2 - (\alpha + \beta) \lesseqgtr 0$
$\alpha + \beta$	$\beta \cancel{>} 0)$	$\alpha + \beta > 0$

$\alpha + \beta \gtrless 2$

Assume that $f\left(\frac{\alpha + \beta}{2}\right) < 0$ so that there are two real roots, and recall that $0 < \alpha < 1$ and $0 < \beta$. If $\alpha + \beta < 2$ both real roots are between 0 and 1 as $f(o)$ is positive and decreasing and $f(1)$ is positive and increasing; hence the

$\alpha + 1$ $f(x)$ $-1-\alpha > 0$ ... < 0 if $\alpha + \beta > 2$ i.e. $\frac{\alpha+\beta}{2} > 1$

minimum point of f(x) and the values of x for which f(x) = o occur between o and 1. By similar reasoning if $\alpha + \beta > 2$ both real roots are between 1 and $\alpha + \beta$. (If $\alpha + \beta = 2$ then $f(\frac{\alpha + \beta}{2}) = f(1) = 1 - \alpha > 0$ so that there are no real roots.) Note that with $o < \alpha < 1$, $\beta > 1$ is a necessary condition for the existence of real roots greater than 1 and $\beta < 1$ is a necessary condition for the existence of real roots less than 1.

$1 \quad . \quad \frac{\alpha+\beta}{2}$

If $f(\frac{\alpha + \beta}{2}) = 0$ there are two identical real roots, these may be less than or greater than 1. (The case of $\alpha = \beta = 1$ yields two real roots equal to 1; this is ruled out by our assumptions.)

If $f(\frac{\alpha + \beta}{2}) > o$ the two roots are conjugate complex numbers. The examination of the properties of these roots will be taken up after solving equation 7.

Solving equation 7 we get that

$\beta = 3$
$\alpha = .9$

$$\mu_1 = \frac{\alpha + \beta + \sqrt{(\alpha+\beta)^2 - 4\beta}}{2} \qquad \mu_2 = \frac{\alpha + \beta - \sqrt{(\alpha+\beta)^2 - 4\beta}}{2} \tag{9}$$

If $(\alpha + \beta)^2 - 4\beta$ (the discriminant) is positive, then there are two real roots; if it is negative, the roots are conjugate complex numbers.

If μ_1 and μ_2 are real numbers (note that $\mu_1 > \mu_2$), then as μ_1 and μ_2 are either both less than or both greater than 1, income is either monotonically increasing ($\mu_1, \mu_2 > 1$) or monotonically decreasing ($\mu_1, \mu_2 < 1$). The income at any date is a weighted average of μ_1 and μ_2 with changing weights. The weight of μ_1 at the t^{th} date is $A_1 \mu_1^{t-1}$ and the weight of μ_2 is $A_2 \mu_2^{t-1}$. As $\mu_1 > \mu_2$ the weight of μ_1 increases in time so that eventually the time series generated behaves as if μ_1 were the only relevant element; μ_1 is called the dominant root.

If μ_1 and μ_2 are conjugate complex numbers then

$$\begin{aligned} \mu_1^t &= \sqrt{\beta}^{\,t} (\cos t\theta + i \sin t\theta) \\ \mu_2^t &= \sqrt{\beta}^{\,t} (\cos t\theta - i \sin t\theta) \end{aligned} \tag{10}$$

where $\sqrt{\beta} = \sqrt{\frac{(\alpha+\beta)^2}{4} + \frac{4\beta - (\alpha+\beta)^2}{4}}$ is the "modulus" and θ is the angle whose cosine equals $\frac{\alpha + \beta}{2\sqrt{\beta}}$ and whose sine equals $\frac{\sqrt{4\beta - (\alpha+\beta)^2}}{2\sqrt{\beta}}$

Equation 5 therefore becomes

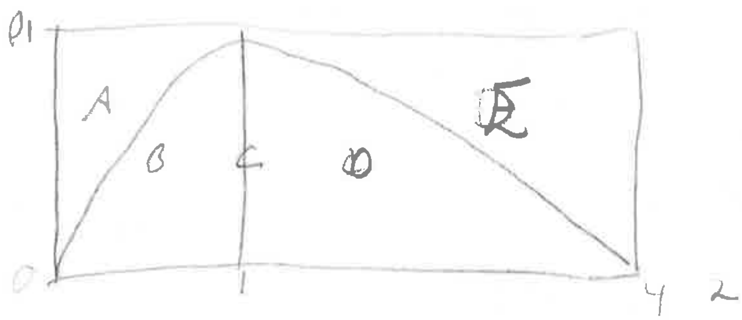
$$Y_t = \sqrt{\beta}^{\,t} (\bar{A}_1 \cos t\theta + \bar{A}_2 \sin t\theta) \text{ where } i \text{ is submerged into } \bar{A}_2. \quad (11)$$

Equation 11 will generate a cycle whose period is $360^\circ/\theta$ and whose amplitude is $2\sqrt{\beta}$. The amplitude of the cycle will decrease in time (the cycle is damped) if $\sqrt{\beta} < 1$, will increase in time (the cycle is explosive) if $\sqrt{\beta} > 1$ or will not change if $\sqrt{\beta} = 1$.

The boundary between cyclical and monotonic time series is given by $(\alpha + \beta)^2 - 4\beta = 0$, and between damped and explosive time series is given by $\beta = 1$. This knowledge as to the types of time series that a linear accelerator-multiplier time series can generate, and the values of α and β which generates each type is exhibited in Diagram A.

Diagram A

The State Diagram (after Samuelson)



Each region and the line $\beta = 1$ of Diagram A represents an alternative possible type of time series that can be generated by the linear model. We will call these alternative types of time series "states" of the economy and depending upon the value of α and β the economy can be in one of five states:

State A: monotonic and damped

" B: cyclical and damped

" C: cyclical and constant amplitude

State D: cyclical and explosive

" E: monotonic and explosive

State A includes the border between states A and B and state E includes the border between states D and E.

For the purposes of business cycle analysis each of these 5 possible states of the economy ~~are~~ is unsatisfactory. States A and D are not cyclical, states B and D while cyclical are either damped or explosive. State E is too regular to represent reality and of course the requirement that $\beta = 1$ is too rigid for acceptance.

In order to use the accelerator-multiplier mechanism to generate more satisfactory types of time series additional specifications have to be made. If the assumed values of α and β are such that the economy is in state A or B, then some regular or random method of sustaining the fluctuations is necessary. If the assumption is that the system is naturally in state D or E, some method of constraining the linear system is required. As an alternative to sustaining, or constraining a linear system, it can be specified that α and β change and their values are either determined by the past of income or are random. Specifying that α and β depend upon past incomes creates a nonlinear model. Another available modification of the simplest linear system is to assume that consumption and investment are not homogeneous functions of income. This is most useful in conjunction with models which require either repeated sustaining or constraining. Of course each time the formal apparatus is modified, openings are made for feeding additional knowledge about the nature of the economy into the model.

So far all that has been exploited is the information contained in α and β and it has been shown that the type of time series generated depends solely upon these parameters. The initial conditions determine the value of A_1 and A_2. The known incomes of two adjacent dates are arbitrarily designated as the income of the zero'th and the first dates, Y_0 and Y_1. If μ_1 and μ_2 are real, from equation 5 we get $Y_0 = A_1 + A_2$ and $Y_1 = A_1 \mu_1 + A_2 \mu_2$ which yields

$$A_1 = \frac{Y_1 - Y_0 \mu_1}{\mu_1 - \mu_2}$$

$$A_2 = \frac{\mu_1 Y_0 - Y_1}{\mu_1 - \mu_2} \qquad (12)$$

If μ_1 and μ_2 are complex, from equation 11 we get $Y_0 = \bar{A}_1 \cos 0^\circ + \bar{A}_2 \sin 0^\circ$ and $Y = \sqrt{\beta}(A_1 \cos \theta^\circ + A_2 \sin \theta^\circ)$ which yields

$$\bar{A}_1 = Y_0$$

$$\bar{A}_2 = \frac{2 Y_1 - (\alpha + \beta) Y_0}{\sqrt{4\beta - (\alpha+\beta)^2}} \qquad (13)$$

The information contained in the initial conditions is of particular importance in determining the behavior of a monotonically explosive series. If $\mu_1 > \mu_2 > 1$, $Y_1 > Y_0 > 0$ but $\mu_2 Y_0 > Y_1$ then $A_1 < 0$, $A_2 > 0$ and $|A_2| > |A_1|$ In this case an initial upward movement of income can result in a downward explosion of income even with μ_1 and μ_2 real and greater than 1.

In what follows the conditions that sustain an otherwise damped series, or constrain an otherwise explosive series will be interpreted as the establishment of new initial conditions.

III Non-linear Models

The specification that α and β are constants, independent of any economic change, can be altered. As is obvious from Table I, rather small changes in α or β can radically change the state of the economy. For example, with β slightly larger than 1.50, a fall in α from .95 to .90 will change the economy from state E to state D. Hence by shifting α or β through rather small ranges a time series is generated which is the result of the economy being in different states at different dates. As the period depends upon α and β, and the amplitude mainly upon β, a time series generated by a variable pattern of changing values of α and β can exhibit a desirable irregularity of both period and amplitude. As any pattern of observations can be reproduced by assuming appropriate values

Table I. Borders Between the States in a Linear Accelerator-Multiplier Model

Values of α	Values of β Boundaries Between Regions		
	A + B	B + D (State C)	~~C +~~ D + E
.95	.60	1	1.50
.90	.47	1	1.73
.80	.31	1	2.09
.70	.20	1	2.40
.60	.14	1	2.66
.50	.09	1	2.91

of α and β, the hypothesis that α and β vary is not satisfactory because it is all encompassing. A more restrictive hypothesis that α and β vary in some specified way is necessary if a refutable and hence acceptable model is to be constructed. Three alternative classes of specifications of how α and β vary can be identified.

(1) α and β can be random variables drawn from some specified population;

(2) α and β can depend upon the past of income;

(3) α and β can depend upon identifiable occurrences exogenous to the formal model.

An accelerator-multiplier model with random α and β coefficients is capable of generating a satisfactory time series. The fact that by appropriately selecting α and β any desired time series can be reproduced is evidence that there exists a population of α's and β's and a draw from this population which can generate an appropriate type of time series. This open-ended existence argument is not sufficiently restrictive to be acceptable. In the random-parameter approach to be useful it is necessary to be able to define the population from which the operative α's and β's are drawn. The population characteristics and the drawing procedure have to be defined as analogous to characteristics of the economy. A beginning toward the studies of such models appears in Minsky (8).

Goodwin (4) studied models in which the accelerator varied both as the result of the past of incomes and as the result of identifiable occurrences exogenous to the model. The formal apparatus for studying these two nonlinear variants is the same.

Goodwin specified that:

(1) there is constant capacity $\overset{*}{K}$ to the capital goods industries;

(2) there is a constant limit $\overset{**}{K}$ to the amount of capital consumption that can occur during any period.

(3) $\beta_t = K$ a constant as long as $\overset{**}{K} < K(Y_{t-1} - Y_{t-2}) < \overset{*}{K}$.

(4) If $K(Y_{t-1} - Y_{t-2}) > \overset{*}{K}$, $\beta_t = \overset{*}{K}/(Y_{t-1} - Y_{t-2})$. $\dfrac{\overset{*}{K}}{Y_{t-1} - Y_{t-2}}$

(5) If $K(Y_{t-1} - Y_{t-2}) < \overset{**}{K}$, $\beta_t = \overset{**}{K}/(Y_{t-1} - Y_{t-2})$. $\dfrac{\overset{**}{K}}{Y_{t-1} - Y_{t-2}}$

Diagram II

Goodwin's relation between β and $Y_{t-1} - Y_{t-2}$

argument is not sufficiently restrictive to be acceptable. In the random-parameter approach to be useful it is necessary to be able to define the population from which the operative α's and β's are drawn. The population characteristics and the drawing procedure have to be defined as analogous to characteristics of the economy. A beginning toward the studies of such models appears in Minsky (8).

Goodwin (4) studied models in which the accelerator varied both as the result of the past of incomes and as the result of identifiable occurrences exogenous to the model. The formal apparatus for studying these two nonlinear variants is the same.

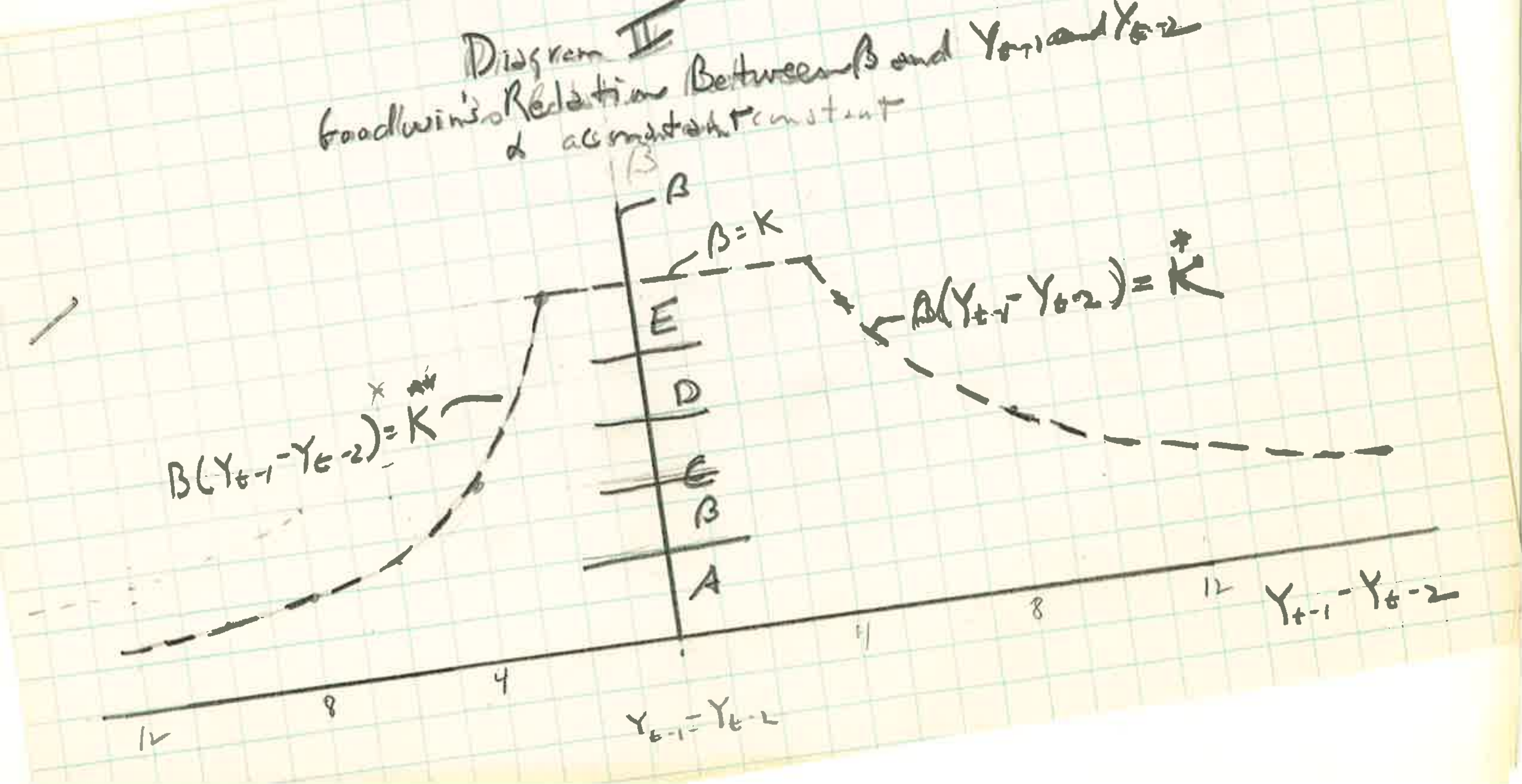
Goodwin specified that:

(1) there is constant capacity $\overset{*}{K}$ to the capital goods industries;

(2) there is a constant limit $\overset{**}{K}$ to the amount of capital consumption that can occur during any period.

(3) $\beta_t = K$ a constant as long as $\overset{**}{K} < K(Y_{t-1} - Y_{t-2}) < \overset{*}{K}$.

(4) If $K(Y_{t-1} - Y_{t-2}) > \overset{*}{K}$ ...



In Goodwin's model, as is illustrated in Diagram II, β is either constant or is a point on a rectangular hyperbola.

Let us assume, with Goodwin, that the unconstrained value of the accelerator coefficient (when $\beta = K$) is sufficiently large so that the economy is in state E. Let us consider increasing incomes. As long as the accelerator coefficient is unconstrained, the monotonic explosive expansion will result in increasing arithmetic increases in income. This will continue until the amount of induced investment exceeds the ceiling to investment, at which time the value of the accelerator falls. As a result future incomes will either be equal to or less than the income associated with the ceiling amount of investment. This in time leads to a monotonic explosive decline in income. A similar turnabout occurs when the disinvestment ceiling becomes effective. As β varies during this process, the period of the cycle being generated will vary.

Although Goodwin specified ceilings and floors to investment as the factor determining the nonlinearity of β, any specification which yields falling accelerator coefficients at large rates of change of income will lead to a similar time series. No matter what the arithmetic change in income may be if, for a given α, β is such that the economy is in state A, income will begin to move monotonically toward its "equilibrium" value. Hence specifications that the financial system is such that the interest rate depends upon the level and the rate of change of income and that high rates of interest when income is rising implies a low β coefficient and that low rates of interest when income is falling also implies a low β coefficient will yield to a time series similar to Goodwin's.

As an alternative set of specifications of the nonlinearity which can be rationalized on economic grounds is:

(1) For small rates of change of income $\beta_t = 0$.

(2) For intermediate rates of change of income $\beta_t = \mathcal{T}(Y_{t-1} - Y_{t-2})$, $\mathcal{T} > 0$.

(3) For large positive rates of change of income $\beta_t = K_1$ which places the economy in state E.

(4) For large negative rates of change of income $\beta_t = K_2$ which places the economy in state A.

The economic assumptions are that small changes in income are not efficient and that large changes are efficient in inducing investment and that a linear function connects the efficient and the inefficient rate of changes of income.

Diagram 3 illustrates how β_t varies with $Y_{t-1} - Y_{t-2}$.

Diagram 3: Alternative Specification of the Non-Linearities

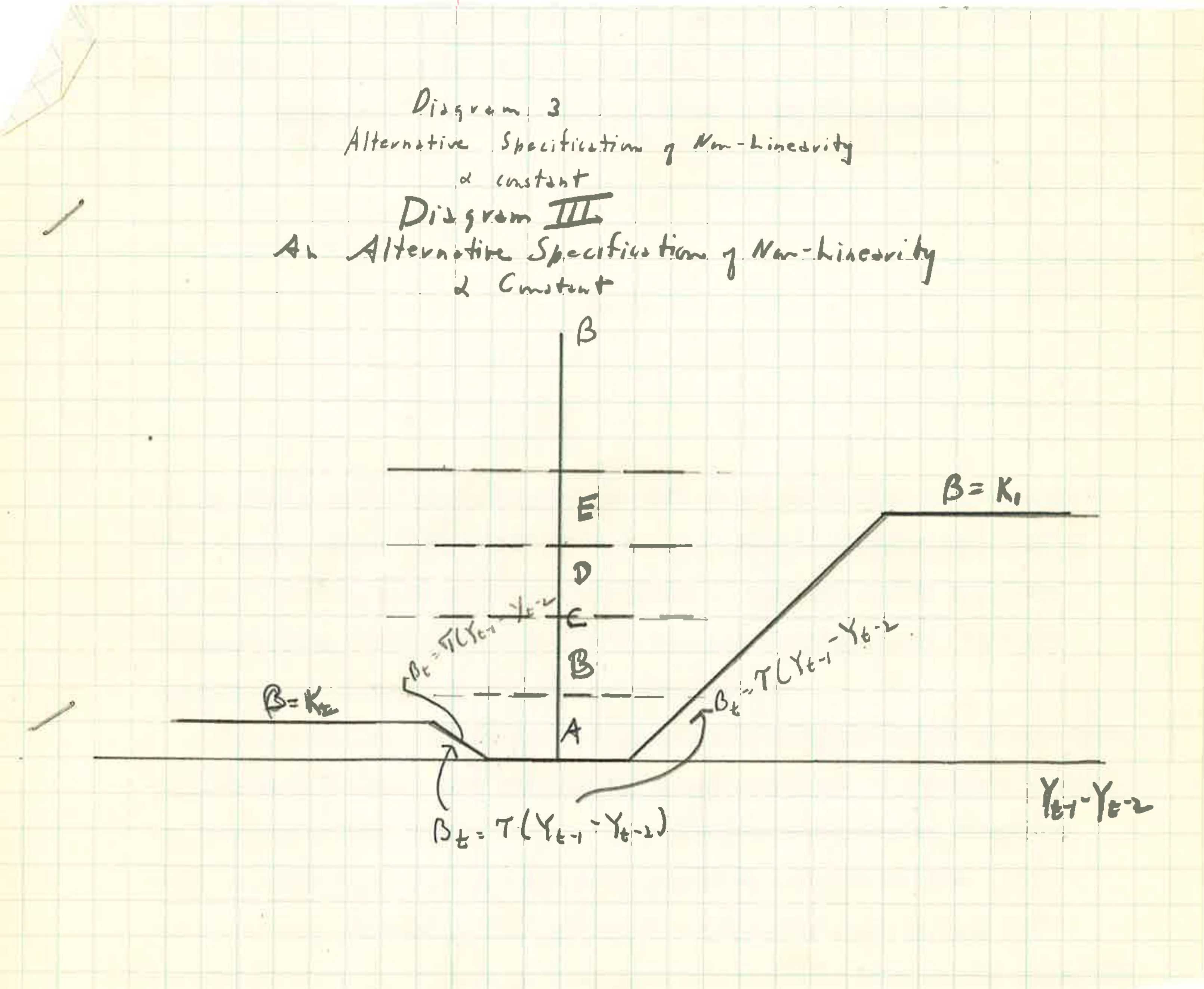
It is obvious that this nonlinear model will either be in state A or in state E. This economy either "stagnates" or "explodes." Such a nonlinear model can be used as a framework for a strong shock theory of business cycles; e.g., the economy will go along as it is unless it receives a strong shock. Wars and financial crises can be interpreted as strong shocks.

The exhibition of a nonlinear model with specifications different from those of Goodwin is evidence that nonlinearity is not sufficient in itself to generate an appropriate type of time series. Goodwin's assumption that an investment ceiling exists which does not change is suspect on economic grounds. If $\overset{*}{K}$ does increase, its rate of increase may be sufficiently great to keep the economy in state E. If this is true, Goodwin's nonlinear case becomes equivalent to the unacceptable state E linear case.

(3) For large positive rates of change of income $\beta_t = K_1$ which places the economy in state E.

(4) For large negative rates of change of income $\beta_t = K_2$ which places the economy in state A.

The economic assumptions are that small changes in income are not efficient and that large changes are efficient in inducing investment and that a linear func-



to the unacceptable state E linear case.

~~III~~ IV. Inherently Damped Linear Models

Consider a linear accelerator-multiplier model with α and β such that the economy is in state B. The cyclical time series generated has decreasing amplitude and a constant period. In time income will settle at some equilibrium income. Given these assumptions, the accelerator-multiplier model can be used only if some mechanism which sustains the cycle exists. Both random shocks to income and systematic injections of income have been suggested as sustaining mechanisms.

The solution equation for a state B time series is

$$Y_t = \beta^{t/2}\left(A_1 \cos t\theta + A_2 \sin t\theta\right);\ 0 < \beta < 1 \text{ and } A_1 = Y_0 \tag{14}$$

$A_2 = \dfrac{2Y_1 - (\alpha+\beta)Y_0}{\sqrt{4\beta - (\alpha+\beta)^2}}$ As in time income will settle at some equilibrium value, equation 14 yields income as a deviation from this equilibrium value. Assume the system is started from this equilibrium value so that $Y_0 = o$ and $Y_1 = C_1 \gtrless 0$. This yields $A_1 = o$ and $A_2 = \dfrac{2C_1}{\quad}$ The solution equation based on these initial conditions is:

$$Y_t = \frac{2\beta^{t/2}}{\sqrt{4\beta-(\alpha+\beta)^2}} C_1 \sin t\theta \tag{15}$$

Equation 15 generates an income in the second period

Actual income deviates from this generated income i.e.,

$$Y_2^{(a)} = Y_2^{(g)} + C_2,\ C_2 \gtrless 0.$$

C_2 is interpreted as the second of two initial conditions for another accelerator-multiplier income generating process; the first initial condition being zero. This yields another particular solution equation

$$Y_t = \frac{2\beta^{t-1/2}}{\sqrt{4\beta-(\alpha+\beta)^2}} C_2 \sin(t-1)\theta \tag{16}$$

As the models are linear, equations 15 and 16 can be added.

By interpreting the deviation of actual income from generated income during each period as the second of two initial conditions (the first initial condition always is zero) for an appropriately dated accelerator-multiplier process we get:

$$Y_n = C_n + \frac{\beta^{1/2}\sin 2\theta}{\sin\theta}C_{n-1} + \frac{\beta^{2/2}\sin 3\theta}{\sin\theta}C_{n-2} + \cdots + \frac{\beta^{\frac{n-1}{2}}\sin n\theta}{\sin\theta}C_1 \quad (17)$$

Income of the nth date is a weighted sum of the deviations of present and past actual incomes from the generated incomes with the accelerator-multiplier process determining the weights. As equation 17 can be written as

$$Y_n = C_n + \frac{\beta^{1/2}}{\sin\theta}\left[\sin 2\theta\, C_{n-1} + \beta^{1/2}\sin 3\theta\, C_{n-2} + \cdots + \beta^{\frac{n-2}{2}}\sin n\theta\, C_1\right] \quad (18)$$

it is obvious that the weight attached to a particular deviation of income from generated incomes decreases with time.

Slutsky (15) showed that a moving average of random shocks will generate a cyclical time series. If the C's are interpreted as random shocks, the time series generated by the accelerator-multiplier model is equivalent to a Slutsky type summation of random shocks time series. The significance of the accelerator-multiplier hypothesis in any random shock-damped model depends upon whether the random shock is merely a "maintaining mechanism" or whether the random elements dominate in the generation of the time series.

Fisher (2) examined a random shock accelerator-multiplier model. He specified that an accelerator-multiplier model with an equilibrium income of 57 billions and $\alpha = .7$, $\beta = .5$ was subject to a random shock drawn from a simulated normal population. The populations mean was zero and standard deviation 5 billions. The resultant time series was not damped and in addition its period and amplitude were variable.

The values of α and β selected by Fisher yield a θ of approximately 30°. With $\beta = .5$, $\theta = 30°$ equation 18 becomes $Y_n = C_n + 1.224\, C_{n-1} + 1.000\, C_{n-2} + .612\, C_{n-3} + .250\, C_{n-4} + 0.0\, C_{n-5} - .125\, C_{n-6} - .153\, C_{n-7} - .125\, C_{n-8} - .0765\, C_{n-9} - .0313\, C_{n-10} + 0.0\, C_{n-11} + .0156\, C_{n-12} +$. The income of any date is primarily determined by recent shocks: in particular the weight attached to shocks more than 5 periods old is small compared to the weight attached to newer shocks. The cycles that Fisher exhibited are mainly the effect of the chance draws of runs of positive or negative shocks and the chance draw of a very large (positive or negative) shock. For example, in a normal distribution with zero mean one fourth of all samples of three drawn at random would be either all positive or all negative. Such not unusual occurrences would "build up" the deviation of Y_n from the equilibrium value.

In a type A, monotonically damped time series deviations of actual from generated incomes are also interpreted as the second initial condition for an accelerator-multiplier solution equation. The resulting solution equation is

$$Y_n = C_n + \left(\frac{\mu_1^2 - \mu_2^2}{\mu_1 - \mu_2}\right) C_{n-1} + \left(\frac{\mu_1^3 - \mu_2^3}{\mu_1 - \mu_2}\right) C_{n-2} + \cdots + \frac{\mu_1^{n-1} - \mu_2^{n-1}}{\mu_1 - \mu_2} C_1 \qquad (19)$$

If $\alpha = .8$ and $\beta = .3$ then $\mu_1 = .6$ and $\mu_2 = .5$. Equation 19 becomes $Y_n = C_n + 1.10\, C_{n-1} + .910\, C_{n-2} + .671\, C_{n-3} + .475\, C_{n-4} + .310\, C_{n-5} + .202\, C_{n-6} + .139\, C_{n-7} + .081\, C_{n-8} + .051\, C_{n-9} + .031\, C_{n-10} + .019\, C_{n-11} + \ldots$ The weights attached to the recent deviations or shocks are much larger than those attached to the earlier shocks. If the C_i are random variables, the luck of the draw will lead to runs of positive or negative shocks. A time series that exhibits the desired cyclical irregularity would be generated. An assumption that the accelerator-multiplier mechanism is inherently cyclical is not necessary in order to obtain the desired type of cycle. The random shock mechanism can generate a cycle even if the accelerator-multiplier mechanism would naturally lead to a monotonic series. Hence the cycle generated by Fisher's model does not depend upon the properties of the accelerator-multiplier mechanism but rather it depends upon the properties of a moving average.

Hansen (6) and Goodwin (3,5) have integrated Schumpeter's innovation hypothesis with the state B accelerator-multiplier model. Even though inventions may occur in a steady stream, the exploitation of inventions and hence the investment due to the innovation process tends to be lumped. Within the framework of equation 17 the C i's are positive during those periods when innovation generated investment is occurring and zero in other periods. Alternative specifications of when in the cycle these lumped innovational investments will occur are 1) immediately after the lower turning point, or 2) when income once again exceeds the equilibrium income. Given that there are k periods to the cycle (determined by the constant α and β coefficients) then this specifies that C_j, $C_j + k$,---, $C_j + n\ k$ (j determined by where in the cycle accumulated innovational investment is injected) will be positive and all other C i's will be zero.

Considerable flexibility can be introduced into this model by allowing the $C_j + n\ k$'s to vary in size and by permitting innovational investment to spill over into the periods following $j + n\ k$. If inventions whose exploitation requires large amounts of investment occur, then $C_j + n\ k$, $C_j + n\ k + 1$ ---- will all be positive and presumably large. This will "build up" a large amplitude and throw the cyclical mechanism out of phase with previous cycles: the innovational shocks dominate the behavior of income. Once these dominating innovations have run their course the accelerator-multiplier mechanism will again determine the path of future income. Innovations which include relatively small amounts of investment will tend to maintain the otherwise damped cycles and when such innovations take place the cycle will exhibit a constant period.

Note also that this innovation shock mechanism will normally tend to produce larger amplitude prosperities than depressions and if the innovational investment occurs at the lower turning point the length of the depression will be shortened.

The accelerator-multiplier hypothesis combined with the innovational hypothesis can generate an appropriately irregular time series. However, this model requires that some mechanism for the accumulation and release of innovational investment exist; and hence a satisfactory economic explanation of the innovation process is necessary.

IV. Inherently Explosive Time Series

The specification that $\beta > 1$ implies that the linear accelerator-multiplier mechanism generates a type D or E time series. Hicks (7) studied the case where this type of time series was constrained by either a ceiling, due to a full employment limit upon productive capacity, or a floor, due to a physical limitation upon the consumption of fixed capital. Note that this model, by considering the ceiling and floor to income explicitly, takes supply conditions into account in determining income. The income determination process is divided into two parts. In the first part income equals aggregate demand as determined by a solution equation of the accelerator-multiplier mechanism. In the second part aggregate supply as given by the ceiling or floor to income determines income.

Consider a monotonically explosive (type E) time series. In the solution equation

$$Y_t^{(s)} = A_1 \mu_1^{\,t} + A_2 \mu_2^{\,t} \quad , \quad \mu_1 > \mu_2 > 1$$

A_1 and A_2 are determined by the initial conditions. At any date t there is a ceiling income C_t and a floor income f_t. The model states that the solution equation will determine income when $f_t \le Y_t^{(s)} \le C_t$ and that if $Y_t > C_t$ then $Y_t = C_t$ and if $Y_t^{(s)} < f_t$ then $Y_t = f_t$

A solution equation with particular values of A_1 and A_2 is a restatement of the content of the accelerator-multiplier process of equation 5 with the two initial incomes known. A deviation of actual income from income as determined by such a solution equation implies that the original solution equation no longer determines future income. This occurs whenever the ceiling or floor to income becomes the effective determinant of actual incomes. Such an occurrence is interpreted as the start of a new accelerator-multiplier process which yields a solution equation that differs from the preceding solution equation only in the values of A_1 and A_2. The initial conditions which determine A_1 and A_2 are the ceiling or floor to income and the income of the date just prior to the ceiling or floor becoming effective. Assuming that the ceiling or floor becomes effective on the n^{th} date, the new solution equation is dated so that $n - 1 = 0$ and $n = 1$.

Let us consider the case of the ceiling becoming the effective determinant of income on the n^{th} date. We have $Y_n = \lambda Y_{n-1}$ and $Y_n < C_n < A_1 \mu_1^n + A_2 \mu_2^n$

From equation 12 we have

$$A_1^1 = \left[\frac{\lambda - \mu_2}{\mu_1 - \mu_2}\right] Y_{n-1} \quad \text{and} \quad A_2^1 = \left(\frac{\mu_1 - \lambda}{\mu_1 - \mu_2}\right) Y_{n-1}$$

As $\mu_1 > \mu_2 > 1$, $A_1^1 > 0$ if $\lambda > \mu_2$ and $A_2^1 > 0$ if $\mu_1 > \lambda$

As shown previously the increase in income as determined by a solution equation is a weighted average of μ_1 and μ_2, hence $\lambda < \mu_1$. This yields $A_2^1 > 0$, but $A_1^1 \geq 0$ or $A_1^1 < 0$ are both possible. If A_1^1 is less than zero then the solution equation with these initial conditions will eventually generate a falling income. With the appropriate initial conditions, an otherwise monotonically explosive time series will generate one turning point.

If $A_1^1 > 0$ then the solution equation will generate a $n+1^{st}$ period's income equal to or greater than λC_n. If the ceiling income increases at some constant rate $\rho \geq 1$ we know that $\lambda > \rho$ Thus if $A_1 \geq 0$ the ceiling income determines actual income for two successive periods. This implies that another new solution equation would operate to generate future incomes. The A's are

$$A_1' = \left[\frac{\rho - \mu_2}{\mu_1 - \mu_2}\right] C_n \quad \text{and} \quad A_2' = \left[\frac{\mu_1 - \rho}{\mu_1 - \mu_2}\right] C_n$$

If $\rho \geq \mu_2$ the ceiling will continue to be the effective determinant of income. As the ceiling, which grows at a constant rate ρ continually determines income the constrained explosive accelerator-multiplier process generates steady growth. If $\rho = \mu_2$ then $A_2'' = 0$ and the accelerator-multiplier process so set in motion degenerates into an unconstrained generator of steady growth. If $\rho < \mu_2$ then $A_2'' < 0$ and $\frac{Y^{(3)}_{n+2}}{Y^{(3)}_{n+1}} < \rho$ income will fall away from the ceiling. In time income will explode downward.

The downward explosion comes to a halt when income as generated by the solution equation would be lower than the floor to income. The analysis in this case is similar to the analysis of what happens at the ceiling. ¶ The solution equation to a type D time series is

$$Y_t = \beta^{t/2}(A_1 \cos t\theta + A_2 \sin t\theta)$$

with $\beta^{t/2} > 1$. If $\rho \geq \beta^{1/2}$ the ceiling to income is not an effective constraint as it increases faster than the amplitude of the explosive cycle. Hence $\beta^{1/2} > \rho$ is the only interesting case.

Assume that the incomes of the dates o and 1 are ceiling incomes, C_o and C_1, and $C_1 = \rho C_0$. With these initial conditions $A_1 = C_0$ and $A_2 = \frac{\rho - \beta^{1/2}\cos\theta}{\beta^{1/2}\sin\theta} C_0$

The solution equation $Y_2^{(3)} = \beta\left(\cos 2\theta + \frac{\rho - \beta^{1/2}\cos\theta}{\beta^{1/2}\sin\theta}\sin 2\theta\right) C_0$ reduces to $Y_2^{(3)} = \left[\rho(\alpha + \beta) - \beta\right] C_0$. If the ceiling is an effective determinant of income in the second period $Y_2^{(3)} > C_2$ or $\rho(\alpha + \beta) - \beta > \rho^2$

Solution of the equation $\rho^2 - \rho(\alpha + \beta) + \beta + K = 0;\ K > 0$

yields $\rho = \dfrac{\alpha + \beta \pm \sqrt{(\alpha + \beta)^2 - 4(\beta + K)}}{2}$, For a type D time series $4\beta > (\alpha + \beta)^2$ hence the rate of growth of ceiling income which would make C_2 the effective determinant of income is complex. As ρ is assumed real, the assumption that the ceiling is a continuing effective determinant of income for the explosive-cyclical case is contradicted. A similar analysis holds for the floor to income. The inherently explosive-cyclical time series will never slide along a ceiling or a floor constraint; it will always bounce off.

If the ceiling income is growing at some exogenously determined rate ρ, and the floor to income depends upon the ceiling income then the amplitude of the cycle that is generated by income bouncing between the floor and the ceiling also grows at the rate ρ. As $\rho > 1$ the "ceiling and floor" cycle becomes equivalent to the explosive accelerator-multiplier model; and with a constant equilibrium income, the relative amplitude of the cycle that is generated increases.

A significant attribute of the constrained explosive accelerator models is that they open the way for the explicit introduction of supply considerations in determining the behavior of income. The constraint need not be interpreted as a full employment production ceiling or a capital consumption production floor. For example, the constraints can be interpreted as attributes of the financial system and the time series generated becomes the result of the interaction of aggregate demand and financial considerations.

VII Nonhomogeneous Models

So far income has been generated as a deviation from some unspecified equilibrium income. To specify the equilibrium income it is sufficient to assume that consumption is a nonhomogeneous function of income, i.e.

$$C_t = \alpha_0 + \alpha_1 Y_{t-1} \qquad 2'$$

Equation 4 becomes

$$Y_t = \alpha_0 + (\alpha_1 + \beta) Y_{t-1} - \beta Y_{t-2} \qquad 4''$$

and the equilibrium income (where $Y_t = Y_{t-1} = Y_{t-2}$) is $\alpha_0 / 1-\alpha_1$. The assumptions as to the behavior of the consumption function associated with Duesenberry (1), Modigliani (9), and Tobin (12) can be used to derive an equilibrium level of income that increases with the ceiling income. As the floor to income is some maximum deviation of income (which varies with time) below the equilibrium income, a nonhomogeneous explosive model does not have to generate an increasing relative amplitude cycle as was true of the homogeneous explosive model. For by specifying that both the equilibrium and the floor to incomes are some (different) constants times the ceiling income a constant relative amplitude cycle can be generated.

In the constrained monotonic explosive case, it was shown that if $\rho \geq \mu_2$ then income will continually press against the ceiling to income whereas if $\rho < \mu_2$ income will bounce off the ceiling to income (ρ is the rate of growth of ceiling income and μ_2 is the smaller root of the solution equation). The economic content of this proposition is that with $\rho \geq \mu_2$ the arithmetic increase in ceiling income induces more than sufficient investment to offset the amount of saving which is determined by the level of income and with $\rho < \mu_2$ the reverse holds true. In the homogeneous case both the ceiling income and the generated income were measured as deviations from the equilibrium income, which implicitly was equal to zero. Once the income generating relation is nonhomogeneous the roots of the solution equation determine the rate of growth of income measured as a deviation from the equilibrium income whereas the rate of growth of ceiling income naturally refers to income measured from zero. As ceiling income is larger than ceiling income minus equilibrium income, it is possible for the amount of investment induced by the arithmetic change in ceiling income to exceed the savings induced by the level of income even if $\mu_2 > \rho$. In this case income will continue to press against the ceiling to income even though the rate of growth of ceiling income seems insufficient to sustain growth.

If the equilibrium income only increases intermittently then the ratio of ceiling income minus equilibrium income to ceiling income will decrease. It follows that after a while the arithmetic increase in ceiling income may be insufficient to offset savings as determined by the achieved level of income. Actual income may press against the ceiling to income for a number of periods and then turn down. Nonhomogeneous explosive models constrained by a ceiling that grows at a constant rate can result in periods of apparent sustained growth that in retrospect become just elongated boom periods of a business cycle.

By interpreting the investment relation as

$$I_t = \beta_0 + \beta_1 (Y_{t-1} - Y_{t-2}) \qquad 3''$$

with β_0 as autonomous or "innovation" investment, equation 4 ~~3~~ becomes

$$Y_t = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) Y_{t-1} - \beta_1 Y_{t-2} \qquad 4''$$

and equilibrium income becomes $\frac{\alpha_0 + \beta_0}{1 - \alpha_1}$. The formal results about constant relative amplitude cycles and the possibility of sustained booms that were derived for the nonhomogeneous case are true when both the investment and the consumption relations are nonhomogeneous. The additional economic content when both are nonhomogeneous is that the equilibrium income is affected by the pace of investment induced by innovations, population growth and shifts etc. It follows that strong bursts of innovational investments can result in lengthy sustained booms for such innovational investments raise the equilibrium level of income relative to the ceiling income.

VI. Conclusions

The examination of the various modifications of the accelerator-multiplier mechanism which have appeared in the literature shows that the apparatus is very flexible. This flexibility makes the apparatus a useful tool for the analysis of income flows. If the accelerator-multiplier mechanism is interpreted as a tool for the analysis of concrete problems in the determination of income rather than as a "theory" of the business cycle, many of the objections to the use of the mechanism vanish. Such an interpretation implies that the α and β coefficients and the behavior of the constraining and the maintaining factors do not have a particular set of values because of characteristics inherent in all economies at all times (or even in one economy at all times). Just as demand and supply curves are useful in the analysis of price determination even though different demand and supply curves have different elasticities, so the accelerator-multiplier analysis as modified by the addition of constraining and sustaining factors is useful in the analysis of income determination.

If the modified accelerator-multiplier approach is to be a useful tool for economic analysis, the freedom of the analyst in choosing the values of the parameters has to be constrained. The two stochastic approaches considered (the stochastic error and stochastic parameter approach) and the nonlinear models which depend upon a variable accelerator coefficient do not seriously constrain the analyst. On the other hand, the linear models with systematic constraining or sustaining factors do limit the analyst's freedom. Of the two types of linear models considered, the inherently damped models plus sustaining factors and the inherently explosive models with added restraining factors, the explosive types seem able to absorb a greater variety of exogenous factors. A particular virtue of the constrained explosive models is that the ceiling and floors can be interpreted as being determined by the financial sectors, thus tying together elements which are often arbitrarily separated in business cycle theory.

To use the explosive cycles with the ceiling and floor constraints it is necessary to estimate the following: 1) the marginal consumption coefficients; 2) the strength of the induced investment reaction; 3) the values and the time rate of change of the autonomous consumption and investment factors; and 4) the nature of the constraints that are effective. This and similar information is often available, and hence the constrained multiplier formulation seems like a useful tool for the analysis of events.

The vitiating element in the argument in favor of the constrained multiplier-accelerator models is that even with all the restrictions upon the nature of constraints that have been suggested, the model may still be so general that all conceivable events are consistent with the hypothesis.